Exercise Sheet 9

Exercise 1. (for credit, due on 23 November) Let X be a connected Riemann surface. We admit that X has a covering by a simply connected Riemann surface $\tilde{X}$. By the uniformization theorem, $\tilde{X}$ is biholomorphic to either $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. We call X an annulus if $\pi_1(X, x_0) \cong \mathbb{Z}$.

(1) *(3 points)* Show that an annulus X is biholomorphic to either $\mathbb{D}^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, $\mathbb{C}\backslash\{0\}$ or $C(a,b) = \{z \in \mathbb{C} \mid a < |z| < b\}$ for some $0 < a < b$.
(2) *(2 points)* Show that $C(a,b)$ is biholomorphic to $C(a',b')$ if and only if $\frac{a}{b} = \frac{a'}{b'}$.

Exercise 2. Let $\Lambda \subset \mathbb{C}$ be a rank-2 lattice. The Weierstrass $\wp$-function is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda\backslash\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

We also define the Eisenstein series

$$g_2 = 60 \sum_{\omega \in \Lambda\backslash\{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda\backslash\{0\}} \frac{1}{\omega^6}.$$

(1) Show that the series defining $\wp$ converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$.
(2) Show that $\wp$ is even, Λ-periodic and has double poles at Λ and no other poles. Show that $\wp'$ is odd, Λ-periodic and has triple poles at Λ and no other poles.
(3) Show that $\wp$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

(4) Let $\Delta := g_2^3 - 27g_3^2$ be the modular discriminant. Show that $4x^3 - g_2x - g_3$ has three distinct roots if and only if $\Delta \neq 0$.

Consider the elliptic curve $E \subset \mathbb{P}^2$ defined by

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3.$$

(5) Check that E is smooth if and only if $\Delta \neq 0$ and that the unique point with $Z = 0$ is $\mathcal{O} = [0 : 1 : 0]$.
(6) Show that the map $\phi : \mathbb{C}/\Lambda \to \mathbb{P}^2$ defined by

$$\phi([z]) = \begin{cases} [\wp(z) : \wp'(z) : 1], & z \notin \Lambda, \\ \mathcal{O}, & z \in \Lambda. \end{cases}$$

is holomorphic on all of $\mathbb{C}/\Lambda$.
(7) Show that $\phi(\mathbb{C}/\Lambda) \subset E$ and that the induced map $\phi_E : \mathbb{C}/\Lambda \to E$ is a biholomorphism. In particular, every complex torus of dimension 1 is algebraic. More is true: every elliptic curves actually arises from some lattice.

Exercise 3. (*) The Lagrangian of a simple pendulum is given by

$$\mathcal{L} = \frac{(\theta')^2}{2} - (1 - \cos\theta),$$

where θ is the angle from the vertical to the pendulum. Make precise the statement: The pendulum's motion is described by a closed periodic orbit on a suitable elliptic curve.